Oscillatory Darcy flow in a horizontal channel containing two non-miscible fluids affected by Lorentz force and thermal radiation

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Abstract: The consequences of thermal radiation on unsteady oscillatory laminar flow of two non-miscible fluids in the vicinity of a porous medium in a horizontal channel are analyzed. The channel is partitioned into phases I and II and both regions are filled with viscous fluids with different physical properties. Phase II contains an electrically conducted fluid which exposed to a homogenous magnetic field. Equations governing the flow problem are partial differential equations (PDEs). Analytical solutions are achieved for the velocity profile along with temperature profile using two-term harmonic and non-harmonic functions. Physical analysis of the significant parameters on velocity along with temperature distributions are shown by means of graphs and explained technically. It is observed that the dampening effects of Darcy's resistance for porous medium lower the velocity of the fluid. The temperature of fluid flowing in the channel is dropped with the rising radiation parameter.

Keywords: Two-phase flow, oscillatory flow, Magnetohydrodynamics, porous medium, thermal radiation, horizontal channel.

Nomenclature	
u_i, v_i	Velocities in x, y directions
x, y, z	Spatial coordinates
t	Time
h	Width of channel
T_i	Temperatures of the fluid

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T_{w_1}, T_{w_2}	Temperatures of the walls of the channel
ΔT	Temperature gradient
K_i	Thermal conductivities of fluids
Cp_i	Specific heat of fluids
q_r	Radiation heat flux
B_0	Magnetic field
v_0	Mean suction velocity
A	Positive constant
$\bar{u}_1$	Average velocity
k	Mean absorption coefficient
k^*	Permeability of porous medium
Ec	Eckert number
$\Pr$	Prandtl number
Rd	Radiation parameter
M	Magnetic parameter
P	Dimensionless pressure gradient
Greek symbols	
μ_i	Dynamic viscosities of fluids
σ	Electrical conductivity of the fluid in phase II
ρ_i	Densities of fluids
φ	Porosity of porous medium
θ_i	Dimensionless temperatures
ω	Frequency parameter
σ^*	Stefan-Boltzmann constant
λ	Porosity parameter
ε	Amplitude of oscillation
α	Ratio of dynamic viscosities
β	Ratio of thermal conductivities
Subscripts	
i	$i=1$ for phase I, $i=2$ for phase II

1. Introduction

Flow of the fluid jointly with heat transfer analysis within a channel through a porous media (A porous media is a volume of material composed of an interconnected void rigid matrix. A porous medium is distinguished not only by its porosity, but also by its permeability) has earned a phenomenal consideration from researchers and engineers for the past couple of decades due to the fact that this research area has significance in many geophysical and engineering applications. These types of applications are water filtration, petroleum engineering, bio-convection in a porous medium, underground waste disposal in geotechnical engineering, scattering of chemical contaminants in water-saturated soil, solid matrix heat exchangers, migration of moisture in fibrous insulation, clean-up of refineries, filter problems in chemical engineering and extraction of geothermal power. The designing of pebble beds reactors in nuclear engineering and thermal reaction between heat-producing porous beds and overlying liquid layer are also examples of this area of research. Heat transfer analysis of unsteady oscillatory flow passing by porous media in a horizontal composite channel with viscous and Darcian dissipation was examined by Umavathi et al. [1]. Singh et al. [2] inspected the transient and non-Darcian impacts on natural convection flow through a vertical channel which was partially saturated with porous media. Cekmer et al. [3] performed the analysis analytically on free convection heat transfer through a channel saturated with a porous medium. Chuhan and Agrawal [4] studied the impact of magnetohydrodynamic convection within a channel which was placed vertically with viscous as well as Ohmic dissipation. Hasnain et al. [5] discussed the impact of porosity in an inclined channel on two non-miscible fluids flow and heat transfer with the magnetic field. Asghar et al. [6] offered the analytical solution of equations represented the problem of heat transfer through porous media within the deformable channel. Using the Adomain decomposition method, Akinshilo [7] study the heat transfer analysis of non-Newtonian fluid between parallel plates with a porous medium. Using the Darcy-Brickman model, Yang et al. [8] investigated the problem of heat transfer as well as entropy generation. They considered N-layers of porous medium which partially filled the channel instead of a single layer.

The flow of an electrically conducted fluid passing through a channel has preeminent importance in several engineering processes due to its potential applications. Designing of cross-field accelerators, magnetohydrodynamic (MHD) generators, shock tubes as well as pumps are some of the applications. Some more applications of MHD channel flow are heat treated substances going among a feed roll and a wind-up roll, glass fiber and aerodynamic extrusion of plastic sheets. Several authors have studied the magnetohydrodynamic fluid flow phenomena in a channel with various assumptions. Malashetty et al. [9] considered the fully developed free convection flow of two magnetohydrodynamic fluids together with heat transfer in a channel and presented approximate solutions using perturbation method. Sivaraj et al. [10] examined mixed convective and laminar flow of two incompressible as well as electrically conducting fluids in a vertical channel. One region of the channel was filled with viscoelastic fluid while other with viscous fluid. Abbas et al. [11] discussed the heat transfer analysis of the MHD flow of two non-miscible fluids through an inclined channel. Furthermore, in another analysis Abbas et al. [12] analyzed the velocity as well as thermal slip impacts on MHD two-phase viscous fluid flow with heat transfer. Akbar et al. [13] numerically studied the analysis of heat and mass transfer within nanofluid flow under the influence of a magnetic field passing through a channel. A channel has both porous walls as well as porous media. In the presence of a magnetic field, Sharma and Mehta [14] examined the unsteady flow of two non-miscible fluids through a horizontal channel with heat transfer. Verma and Gupta [15] examined the MHD viscous fluid flow passing through the porous channel with suction/injection. VeeraKrishna and Chamkha [16] studied the MHD unsteady flow of non-Newtonian fluid within infinite vertical plates filled with porous media. They also take into account the outcomes of radiation and chemically reactive species.

In the above studies analysis of radiative heat transfer in natural convection has not been considered, however, radiative-convective flows come across in a variety of technological and environmental procedures such as in fire research, cooling and heating of channels, aeronautics and so on. Therefore, the study of convective along with thermal radiative flow has great importance due to their association in several practical applications. Chauhan and Kumar [17] did an analysis of fully developed mixed convection viscous fluid flow with thermal radiation and viscous dissipation between two infinite vertical parallel walls. Mathew et al. [18] looked into the problem of the combined influence of both radiation and Hall current. In this investigation the authors, considered convective heat and mass transfer within the MHD fluid flow through rotating horizontal porous channel. Heat and mass transfer analysis was carried out by Hayat et al. [19] of the three-dimensional boundary layer flow of MHD viscous fluid within two infinite parallel plates with radiation. Between the vertical porous plate, Prakash and Muthtamilselvan [20] analyzed the effect of thermal radiation on MHD flow of micropolar fluid. Degan et al. [21] discussed the impact of hydrodynamic anisotropy over mixed convection through vertical channel filled with porous medium and affected with radiation. The influence of Hall current and radiation on the MHD oscillatory flow passing through a vertical channel in the presence of porous media was investigated by VeeraKrishna et al. [22].

Because of the complex structure of porous media, the liquid flowing by it is jumbled in a chaotic manner that significantly improves the heat transfer among liquid and the solid segment comprising the media. Due to this, porous media is chosen in various heat transfer regions including stirling engine, cryocoolers and regenerative heat exchanger. Therefore, the main purpose of the present investigation is to examine the oscillatory flow of two immiscible viscous fluids within a porous space channel which is placed horizontally with the magnetic field and radiation. An approximate analytical solution of coupled partial differential equations is developed for the velocity and temperature distributions by employing two-term harmonic and non-harmonic functions. Outcomes of some important physical parameters on the fluid velocity along with temperature are examined and discussed through tables and graphs.

2. Problem formulation

Consider a fully developed and laminar flow of two immiscible fluids flowing through a horizontal channel extending in the $Z-$ and $X-$directions. Geometrically, the problem under discussion is depicted in Fig. 1. The region $0 \le y \le h$ is named as phase I and the region $-h \le y \le 0$ is labeled as phase II. Phase I is occupied with a viscous fluid and phase II is filled by an electrically conducting viscous fluid. An external magnetic field of homogeneous intensity B_0 is applied in the $y-$ direction and the flow is confined in a porous medium. The walls of the channel are assumed to be at constant temperatures T_{w_1} and T_{w_2} with the condition $T_{w_1} > T_{w_2}$. The characteristics of fluid like viscosities, densities, and conductivities of both fluids are supposed to be different and constant. The flow in a channel is controlled by a constantly applied pressure gradient $(-\partial p/\partial y)$ and temperature gradient i.e. $\Delta T = T_{w_1} - T_{w_2}$.

Under these suppositions and considering $\rho_1 = \rho_2 = \rho_0$ and $Cp_1 = Cp_2 = Cp,$ the flow equations for both phases are presented in the following form

For phase-I

$$\frac{\partial v_1}{\partial y} = 0, \tag{1}$$

$$\rho_0\left(\frac{\partial u_1}{\partial t} + v_1\frac{\partial u_1}{\partial y}\right) = \mu_1\frac{\partial^2 u_1}{\partial y^2} + \frac{\partial p}{\partial x} - \frac{\varphi\mu_1}{k^*}u_1, \tag{2}$$

$$\rho_0 Cp\left(\frac{\partial T_1}{\partial t} + v_1\frac{\partial T_1}{\partial y}\right) = K_1\frac{\partial^2 T_1}{\partial y^2} + \mu_1\left(\frac{\partial u_1}{\partial y}\right)^2 - \frac{\partial q_r}{\partial y}, \tag{3}$$

For phase-II

$$\frac{\partial v_2}{\partial y} = 0, \tag{4}$$

$$\rho_0\left(\frac{\partial u_2}{\partial t} + v_2\frac{\partial u_2}{\partial y}\right) = \mu_2\frac{\partial^2 u_2}{\partial y^2} - \frac{\partial p}{\partial x} - \frac{\varphi\mu_2}{k^*}u_2 - \sigma B_0^2 u_2, \tag{5}$$

$$\rho_0 Cp\left(\frac{\partial T_2}{\partial t} + v_2\frac{\partial T_2}{\partial y}\right) = K_2\frac{\partial^2 T_2}{\partial y^2} + \mu_2\left(\frac{\partial u_2}{\partial y}\right)^2 - \frac{\partial q_r}{\partial y} + \sigma B_0^2 u_2^2, \tag{6}$$

By applying Rosseland approximation [23], the radiative heat flux is explained as follows

$$q_r = -\frac{4\sigma^*}{3k}\frac{\partial T^4}{\partial y}, \tag{7}$$

T^4 is stated in the form of a linear function of the temperature by employing Taylor's series about T_{w_2} and hence we get

$$T^4 = 4T_{w_2}{}^3 T - 3T_{w_2}{}^4. \tag{8}$$

The fluid satisfies the no-slip boundary condition on the walls of the channel. Furthermore, the continuity of fluid velocity, fluid temperature as well as shear stress is expected at the interface region. Hence the appropriate boundary and interface conditions for the velocity profiles are

$$u_1(h) = 0, \quad u_2(-h) = 0, \quad u_1(0) = u_2(0),$$
$$\mu_1 \frac{\partial u_1}{\partial y} = \mu_2 \frac{\partial u_2}{\partial y} \text{ at } y = 0, \tag{9}$$

and for the temperature are

$$T_1(h) = T_{w_1}, \quad T_2(-h) = T_{w_2}, \quad T_1(0) = T_2(0),$$
$$K_1 \frac{\partial T_1}{\partial y} = K_2 \frac{\partial T_2}{\partial y} \text{ at } y = 0. \tag{10}$$

Following Chamkha [24], we are taking (assuming $v_1 = v_2 = v$)

$$v = v_0(1 + \varepsilon A e^{i\omega t}), \tag{11}$$

here A symbolize positive constant with $\varepsilon << 1$ such that $\varepsilon A \leq 1$. The transpiration velocity is assumed to vary periodically with time about a non-zero constant mean suction velocity v_0 and the case of constant transpiration velocity can be obtained by taking $\varepsilon A = 0$.

The following non-dimensional variables and quantities are applied to reduce the governing equations in non-dimensional form

$$u_i = u_i^* \bar{u}_1, \quad y = y^* h, \quad t = \frac{h^2}{\nu} t^*, \; v = \frac{\nu}{h} v^* = \frac{\nu}{|v_0|}, \quad \omega = \frac{\nu}{h^2}\omega^*, \theta = \frac{T - T_{w_2}}{T_{w_1} - T_{w_2}},$$
$$P = \frac{h^2}{\mu_1 \bar{u}_1}\left(\frac{\partial p}{\partial x}\right), \lambda = h^2 \varphi / k^*, Ec = \bar{u}_1^2 / C_p \Delta T, Pr = \mu_1 C_p / K_1, \beta = K_2 / K_1, \tag{12}$$
$$Rd = 16\sigma^* T_{w_2}{}^3 / 3kK_1, M = B_0 h\sqrt{\sigma / \mu_2}, \alpha = \mu_2 / \mu_1,$$

The asterisks are removed for convenience with the assumption that now all the quantities are dimensionless, so Eqs. (2) – (6) become

For Phase-I

$$\frac{\partial u_1}{\partial t} + v\frac{\partial u_1}{\partial y} = \frac{\partial^2 u_1}{\partial y^2} - P - \lambda u_1, \tag{13}$$

$$\frac{\partial\theta_1}{\partial t}+v\frac{\partial\theta_1}{\partial y}=\left(\frac{1+Rd}{\Pr}\right)\frac{\partial^2\theta_1}{\partial y^2}+Ec\left(\frac{\partial u_1}{\partial y}\right)^2, \tag{14}$$

For Phase-II

$$\frac{\partial u_2}{\partial t}+v\frac{\partial u_2}{\partial y}=\alpha\frac{\partial^2 u_2}{\partial y^2}-P-\lambda\alpha u_1-M^2\alpha u_2, \tag{15}$$

$$\frac{\partial\theta_2}{\partial t}+v\frac{\partial\theta_2}{\partial y}=\left(\frac{\beta+Rd}{\Pr}\right)\frac{\partial^2\theta_1}{\partial y^2}+\alpha Ec\left(\frac{\partial u_1}{\partial y}\right)^2+EcM^2\alpha u_2{}^2. \tag{16}$$

The boundary conditions in new variables become

$$u_1(1)=0,\quad u_2(-1)=0,\quad u_1(0)=u_2(0),$$
$$\frac{\partial u_1}{\partial y}=\alpha\frac{\partial u_2}{\partial y}\text{ at } y=0, \tag{17}$$
$$\theta_1(1)=1,\quad \theta_2(-1)=0,\quad \theta_1(0)=\theta_2(0),$$
$$\frac{\partial\theta_1}{\partial y}=\beta\frac{\partial\theta_2}{\partial y}\text{ at } y=0, \tag{18}$$

3. Solution methodology

The equations (13)-(16) are solved with the conditions (17) and (18). These equations are coupled nonlinear PDEs and their closed-form solution cannot be achieved. However, these governing equations of the flow can be reduced to ordinary differential equations by assuming

$$u_1(y,t)=u_{10}(y)+\varepsilon e^{i\omega t}u_{11}(y),\ u_2(y,t)=u_{20}(y)+\varepsilon e^{i\omega t}u_{21}(y), \tag{19}$$
$$\theta_1(y,t)=\theta_{10}(y)+\varepsilon e^{i\omega t}\theta_{11}(y),\ \theta_2(y,t)=\theta_{20}(y)+\varepsilon e^{i\omega t}\theta_{21}(y). \tag{20}$$

This is a valid assumption because of the choice of v as defined in Eq. (11) that the amplitude $\varepsilon A<<1$.

Considering the real part of $e^{i\omega t}$, Eqs. (19) and (20) become

$$u_1(y,t)=u_{10}(y)+\varepsilon\cos\omega t\, u_{11}(y),\ u_2(y,t)=u_{20}(y)+\varepsilon\cos\omega t\, u_{21}(y), \tag{21}$$
$$\theta_1(y,t)=\theta_{10}(y)+\varepsilon\cos\omega t\,\theta_{11}(y),\ \theta_2(y,t)=\theta_{20}(y)+\varepsilon\cos\omega t\,\theta_{21}(y). \tag{22}$$

Using Eqs. (21) - (22) in Eqs. (13) - (16), after comparing harmonic and non-harmonic terms and neglecting the terms of $O\left(\varepsilon^2\right)$, we obtain the following system of equations.

For Phase-I

Non-periodic coefficients

$$\frac{d^2u_{10}}{dy^2}-\frac{du_{10}}{dy}-\lambda u_{10}=P, \tag{23}$$

$$\left(1+Rd\right)\frac{d^2\theta_{10}}{dy^2}-\Pr\frac{d\theta_{10}}{dy}=-\Pr Ec\left(\frac{du_{10}}{dy}\right)^2. \tag{24}$$

Periodic coefficients

$$\frac{d^2u_{11}}{dy^2}-\frac{du_{11}}{dy}+\omega\tan\omega t\,u_{11}-\lambda u_{11}=A\frac{du_{10}}{dy}, \tag{25}$$

$$\left(1+Rd\right)\frac{d^2\theta_{11}}{dy^2}-\Pr\frac{d\theta_{11}}{dy}+\Pr\omega\tan\omega t\theta_{11}=\Pr A\frac{d\theta_{10}}{dy}-2\Pr Ec\frac{du_{10}}{dy}\frac{du_{11}}{dy}. \tag{26}$$

For Phase-II

Non-periodic coefficients

$$\alpha\frac{d^2u_{20}}{dy^2}-\frac{du_{20}}{dy}-\alpha\lambda u_{20}-\alpha M^2u_{20}=P, \tag{27}$$

$$\left(\beta+Rd\right)\frac{d^2\theta_{20}}{dy^2}-\Pr\frac{d\theta_{20}}{dy}=-\Pr Ec\alpha\left(\frac{du_{20}}{dy}\right)^2-\Pr Ec\alpha M^2{u_{20}}^2. \tag{28}$$

Periodic coefficients

$$\alpha\frac{d^2u_{21}}{dy^2}-\frac{du_{21}}{dy}+\omega\tan\omega t\,u_{21}-\alpha\lambda u_{21}-\alpha M^2u_{21}=A\frac{du_{20}}{dy}, \tag{29}$$

$$\left(\beta+Rd\right)\frac{d^2\theta_{21}}{dy^2}-\Pr\frac{d\theta_{21}}{dy}+\Pr\omega\tan\omega t\theta_{21}=\begin{bmatrix}\Pr A\frac{d\theta_{20}}{dy}-2\alpha\Pr Ec\frac{du_{20}}{dy}\frac{du_{21}}{dy}\\-2\alpha\Pr EcM^2u_{20}u_{21}\end{bmatrix}. \tag{30}$$

The corresponding boundary conditions (17) and (18) reduce to

Non-periodic coefficients

$$u_{10}\left(1\right)=0,\quad u_{20}\left(-1\right)=0,\quad u_{10}\left(0\right)=u_{20}\left(0\right),$$
$$\frac{\partial u_{10}}{\partial y}=\alpha\frac{\partial u_{20}}{\partial y}\ \text{ at } y=0. \tag{31}$$

Periodic coefficients

$$u_{11}(1)=0, \quad u_{21}(-1)=0, \quad u_{11}(0)=u_{21}(0),$$
$$\frac{\partial u_{11}}{\partial y}=\alpha\frac{\partial u_{21}}{\partial y} \text{ at } y=0. \tag{32}$$

Non-periodic coefficients

$$\theta_{10}(1)=1, \quad \theta_{20}(-1)=0, \quad \theta_{10}(0)=\theta_{20}(0),$$
$$\frac{\partial \theta_{10}}{\partial y}=\beta\frac{\partial \theta_{20}}{\partial y} \text{ at } y=0. \tag{33}$$

Periodic coefficients

$$\theta_{11}(1)=0, \quad \theta_{21}(-1)=0, \quad \theta_{11}(0)=\theta_{21}(0),$$
$$\frac{\partial \theta_{11}}{\partial y}=\beta\frac{\partial \theta_{21}}{\partial y} \text{ at } y=0. \tag{34}$$

Eqs. (23) – (30) are the ordinary differential equations along with the boundary and interface conditions (31) – (34) which can be solved in closed form.

The periodic terms correspond to the steady flow for both fluids and the steady-state solutions for both the velocity and temperature profiles can be expressed as

$$u_{10}=-\frac{P}{\lambda}+C_1e^{m_1y}+C_2e^{m_1y}, \tag{35}$$

$$u_{20}=-\frac{P}{\alpha m_3}+C_3e^{m_4y}+C_4e^{m_5y}, \tag{36}$$

$$\theta_{10}=C_5+C_6\,e^{m_{12}y}+k_1e^{2m_1y}+k_2e^{2m_2y}+k_3e^{m_{11}y}. \tag{37}$$

$$\theta_{20}=C_7+C_8\,e^{m_{14}y}+k_4y+k_5e^{m_4y}+k_6e^{m_5y}+k_7e^{2m_4y}+k_8e^{2m_5y}+k_9e^{m_{13}y}, \tag{38}$$

However, the periodic (non-harmonic) terms solutions or transient velocity and temperature distributions in both phases of the channel (Phase-I and Phase-II) carry out various forms depending on the value of $4\omega\tan\omega t$ for both fluid velocity and temperature. These types of forms can be presented to be the following ones.

Case I

Phase I

$$u_{11}=C_9e^{m_{15}y}+C_{10}e^{m_{16}y}+k_{12}e^{m_1y}+k_{13}e^{m_2y} \tag{39}$$
$$\text{for} \quad 4(\omega\tan\omega t-\lambda)<1,$$

$$\theta_{11} = C_{13}\, e^{m_{19}y} + C_{14}\, e^{m_{20}y} + k_{17}e^{m_{12}y} + k_{18}e^{m_{11}y} + k_{19}e^{2m_1 y} + k_{20}e^{2m_2 y} + k_{21}e^{m_{21}y} + k_{22}e^{m_{22}y} + k_{23}e^{m_{23}y} + k_{24}e^{m_{24}y} \quad (40)$$

for $4\omega \tan \omega t(1+Rd) < \Pr$.

Phase II

$$u_{21} = C_{11}e^{m_{17}y} + C_{12}e^{m_{18}y} + k_{15}e^{m_4 y} + k_{16}e^{m_5 y} \quad (41)$$

for $4\alpha(\omega \tan \omega t - \alpha(M^2+\lambda)) < 1,$

$$\theta_{21} = C_{15}\, e^{m_{25}y} + C_{16}\, e^{m_{26}y} + k_{25} + k_{26}e^{m_4 y} + k_{27}e^{m_5 y} + k_{28}e^{2m_4 y} + k_{29}e^{2m_5 y} + k_{30}e^{m_{14}y} + k_{31}e^{m_{13}y} + k_{32}e^{m_{17}y} + k_{33}e^{m_{18}y} + k_{34}e^{m_{27}y} + k_{35}e^{m_{28}y} + k_{36}e^{m_{29}y} + k_{37}e^{m_{30}y} \quad (42)$$

for $4\omega \tan \omega t(\beta + Rd) < \Pr$.

Case II

Phase I

$$u_{11} = (D_1 + D_2 y)e^{J_1 y} + k_{12}e^{m_1 y} + k_{13}e^{m_2 y}$$

(43)

for $4(\omega \tan \omega t - \lambda) = 1,$

$$\theta_{11} = (D_5 + D_6 y)e^{J_3 y} + k_{17}e^{m_{12}y} + k_{18}e^{m_{11}y} + k_{19}e^{2m_1 y} + k_{20}e^{2m_2 y} + P_{14}e^{n_1 y} + P_{15}e^{n_2 y} + P_{16}ye^{n_1 y} + P_{17}ye^{n_2 y} \quad (44)$$

for $4\omega \tan \omega t(1+Rd) = \Pr$.

Phase II

$$u_{21} = (D_3 + D_4 y)e^{J_2 y} + k_{15}e^{m_4 y} + k_{16}e^{m_5 y} \quad (45)$$

for $4\alpha(\omega \tan \omega t - \alpha(M^2+\lambda)) = 1,$

$$\theta_{21} = (D_7 + D_8 y)e^{J_4 y} + k_{25} + k_{26}e^{m_4 y} + k_{27}e^{m_5 y} + k_{28}e^{2m_4 y} + k_{29}e^{2m_5 y} + k_{30}e^{m_{14}y} + k_{31}e^{m_{13}y} + P_{26}e^{J_1 y} + P_{27}e^{J_2 y} + P_{28}e^{n_3 y} + P_{29}e^{n_4 y} + P_{30}ye^{n_3 y} + P_{31}ye^{n_4 y} \quad (46)$$

for $4\omega \tan \omega t(\beta + Rd) = \Pr$.

Case III

Phase I

$$u_{11} = (G_1 \cos \delta_1 y + G_2 \sin \delta_1 y)e^{\gamma_1 y} + k_{12}e^{m_1 y} + k_{13}e^{m_2 y} \quad (47)$$

for $4(\omega \tan \omega t - \lambda) > 1,$

$$\theta_{11} = (G_5 \cos \delta_3 y + G_6 \sin \delta_3 y)e^{\gamma_3 y} + k_{17}e^{m_{12}y} + k_{18}e^{m_{11}y} + k_{19}e^{2m_1 y} + k_{20}e^{2m_2 y} + (R_{18} \cos \delta_1 y + R_{19} \sin \delta_1 y)R_{16}e^{w_1 y} + (R_{20} \cos \delta_1 y + R_{21} \sin \delta_1 y)R_{17}e^{w_2 y} \quad (48)$$

for $4\omega \tan \omega t(1+Rd) > \Pr$.

Phase II

$$u_{21} = (G_3 \cos \delta_2 y + G_4 \sin \delta_2 y)e^{\gamma_2 y} + k_{15}e^{m_4 y} + k_{16}e^{m_5 y} \quad (49)$$

$$\text{for} \quad 4\alpha(\omega\tan\omega t-\alpha(M^2+\lambda))>1,$$

$$\begin{aligned}\theta_{21}&=(G_7\cos\delta_4 y+G_8\sin\delta_4 y)e^{\gamma_4 y}+k_{25}+k_{26}e^{m_4 y}+k_{27}e^{m_5 y}+k_{28}e^{2m_4 y}\\&+k_{29}e^{2m_5 y}+k_{30}e^{m_{14} y}+k_{31}e^{m_{13} y}+(R_{37}\cos\delta_2 y+R_{38}\sin\delta_2 y)R_{30}e^{\gamma_2 y}\\&+(R_{39}\cos\delta_2 y+R_{40}\sin\delta_2 y)e^{w_3 y}+(R_{41}\cos\delta_2 y+R_{42}\sin\delta_2 y)e^{w_4 y}\\&\text{for} \quad 4\omega\tan\omega t(\beta+Rd)>\Pr.\end{aligned} \tag{50}$$

All constants arising in the above equations are expressed in the Appendix section.

4. Results and discussion

We computed the velocity $u(y)$ and temperature distributions $\theta(y)$ by solving the coupled PDEs analytically using the two-term harmonic and non-harmonic functions as shown in Eqs. (19) and (20). The velocity and temperature profiles of the two immiscible fluids in both phases of the channel are plotted and discussed for various parametric conditions. Figs. 2-6 are presented to show the influence of M, λ and Rd for the three different cases (Case I, II, III) as mentioned above. Tables 1-6 are tabulated to show the effects of the amplitude εA and the periodic frequency ωt on the velocity and temperature profiles of both fluids in the channel.

Fig. 2 is presented to show the effects of the M on $u(y)$ in the channel for three different cases. Since the fluid in the lower phase (Phase II) of the channel is electrically conducting and is affected by the magnetic field, so $u(y)$ for fluid flowing in Phase II is decreasing significantly as compared to the fluid in Phase I. The decrement is due to an increase in magnetic field which is applied perpendicular to the flow direction of fluid. The applied magnetic field increases the resistive force named as the Lorentz force which results in the reduction of velocity within the channel for all three cases. The influence of M on $\theta(y)$ is shown in Fig. 3. It can be seen clearly in the figure that the temperature of an electrically-conducting fluid flowing in Phase II is increasing with the increasing values of the M. The resistive force within the flow increases as the magnetic field increases. As shown in Fig., this tends to raise the fluid temperature for all cases I, II and III.

Porosity refers to the ratio of the void volume to the total volume of the substance, therefore to see the usefulness of such material in the flow, Fig. 4 and 5 are drawn. Fig. 4 elucidates the effects of λ on $u(y)$ of both fluids in the channel. The damping impacts of the Darcy resistance cause fluid speed to reduce across the entire channel (for all cases). This expected phenomenon is because physically, an increase in the porosity of the medium enhances the flow obstruction and thus, the flow rate will be decreased. Fig. 5 displays the influence of λ on $\theta(y)$. The effect of λ on $\theta(y)$ is the same as its effects on the velocity of the fluid.

The decreasing effect of Rd on $\theta(y)$ can be observed from Fig. 6. The cause for this decrease is the fact that the impact of radiation suppresses the effect of the natural convection by dropping the temperature variance between the fluid and the walls of the channel, hence the temperature decreases. The reason for this drop is that radiation's impact suppresses the effect of natural convection by lowering the variation in temperature between the liquid and the channel walls and consequently reduces the temperature for all cases.

Tables 1-6 exhibit the influence of the amplitude εA and the periodic frequency parameter ωt on u and θ in both phases of the channel for the three different cases. Tables 1 and 2 show these effects for Case I; Tables 3 and 4 for Case II and Tables 5 and 6 for Case III. It can be noticed from the tables that u increases as εA increases for Cases I and III whereas it decreases for Case II. The impact of the increase in εA on θ is the same as its impact on u which can be observed in Tables 1, 3, 5.

The effects of ωt on the flow field for the three different cases of solutions are presented in Tables 2, 4 and 6. For Case I (Table 2), the fluid velocity remains almost constant for change in ωt while θ increases with the rising values of ωt. The fluid velocity through the channel remains almost the same for Case III (Table 6) however θ reduces with the raise in ωt. Table 4 is tabulated to show the effects of ωt on u and θ for Case II. This table shows that both u and θ decrease slightly with an increase in the frequency parameter.

5. Conclusions

This paper considers the MHD oscillatory flow of two immiscible fluids in a horizontal porous media channel exposed to the thermal radiation. The two-term harmonic and non-harmonic functions have been used to obtain the solutions of the governing flow equations. The impacts of the pertinent parameters on the u and θ were discussed through graphs and tables whose summary is as follows

- Both u and θ decrease with increasing values of the porosity parameter λ.
- The fluid velocity decreases with a rise in M whilst the temperature of fluid increases as the magnetic parameter increases.
- The fluid temperature for both phases of the channel decreases with increasing values of the radiation parameter Rd.
- The amplitude of the transpiration velocity has a significant impact on the flow and heat transfer aspects of fluids flowing in the channel.
- From tabulation, it can be found that u increases as amplitude εA increases for Cases I and III, while for Case II it decreases.

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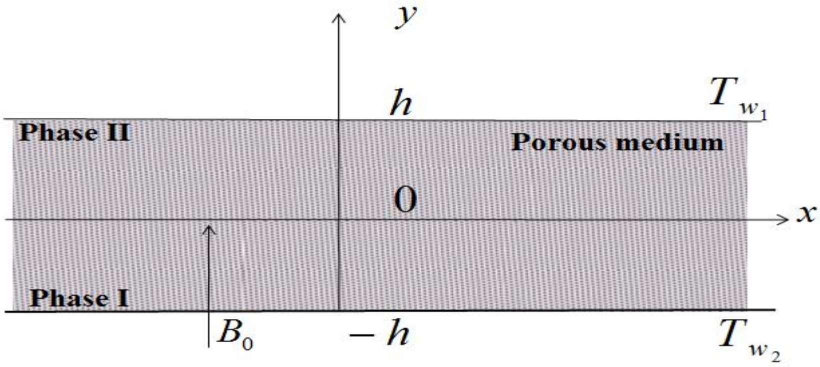

Fig. 1: Physical configuration of the problem

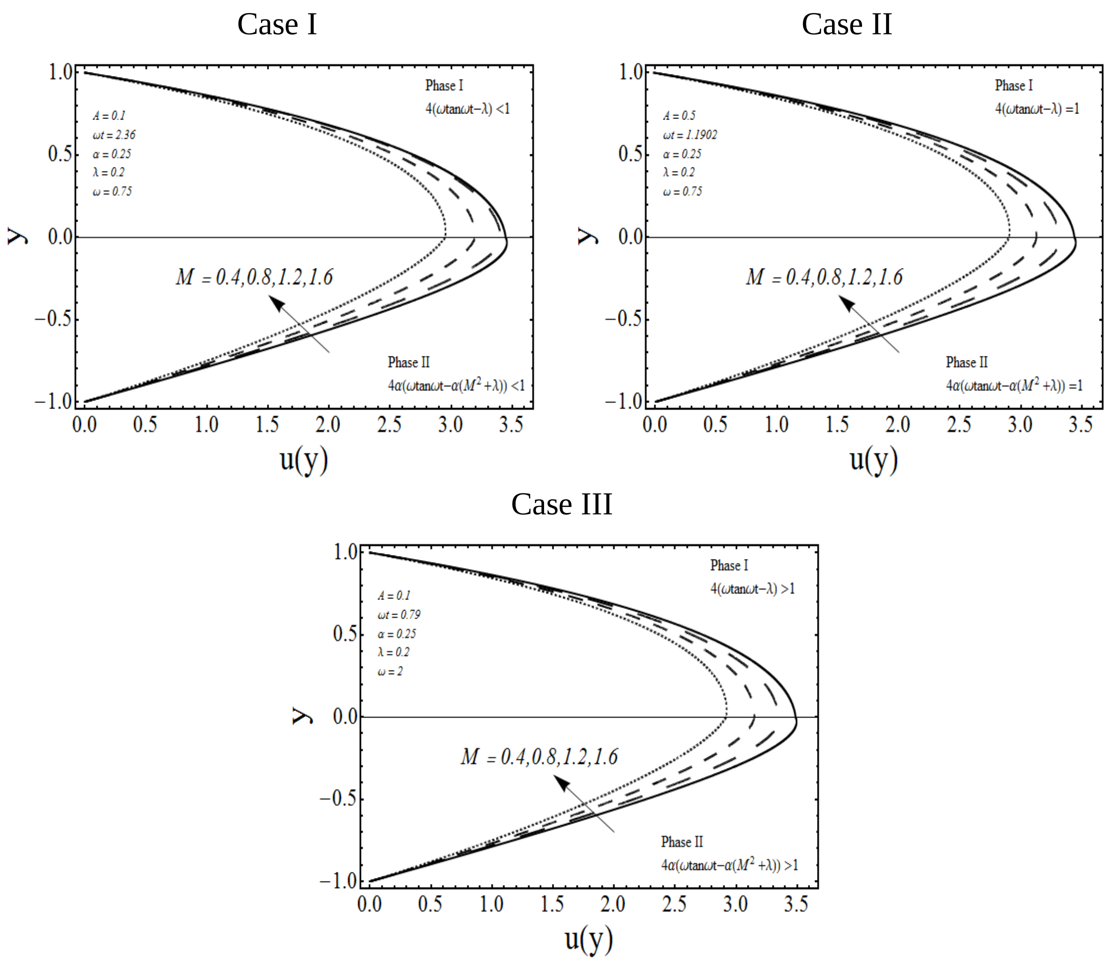

Fig. 2: $u(y)$ for M.

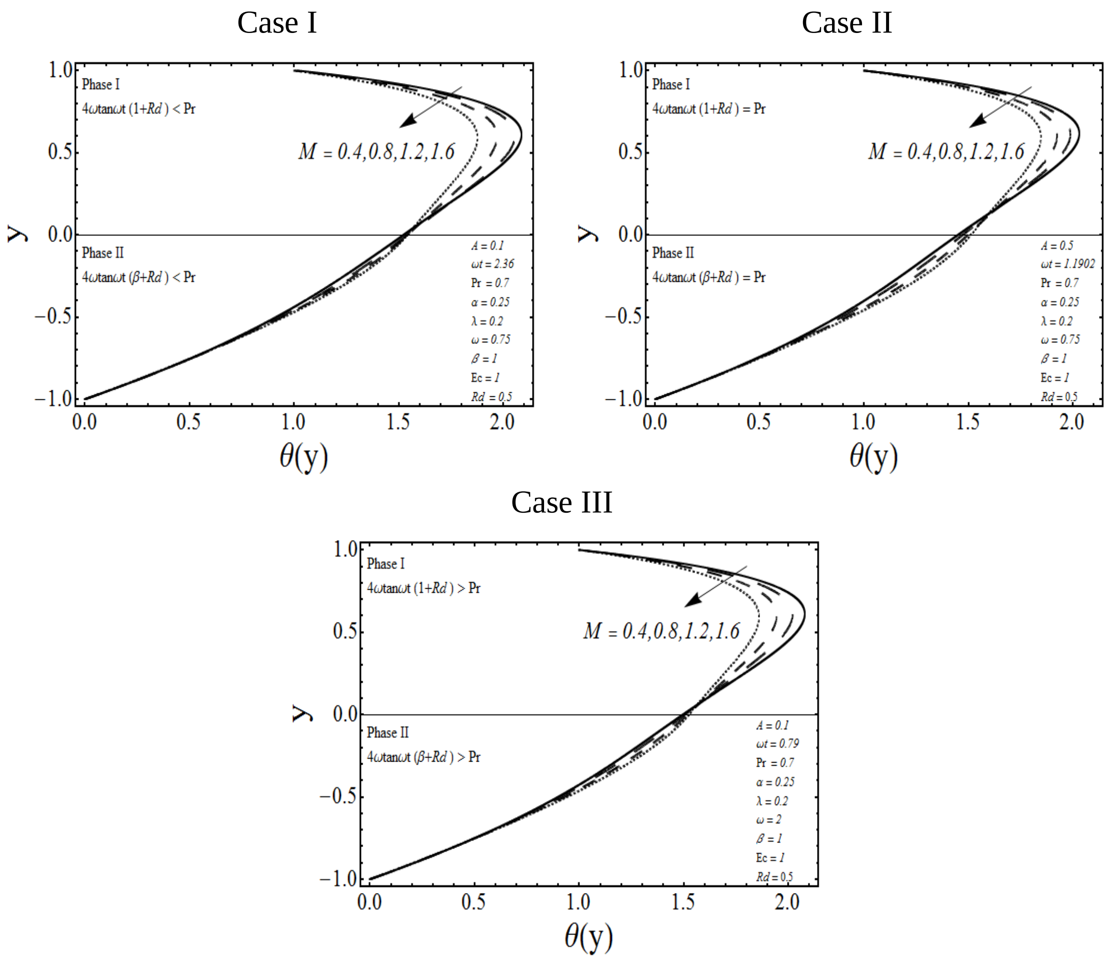

Fig. 3: $\theta(y)$ for M.

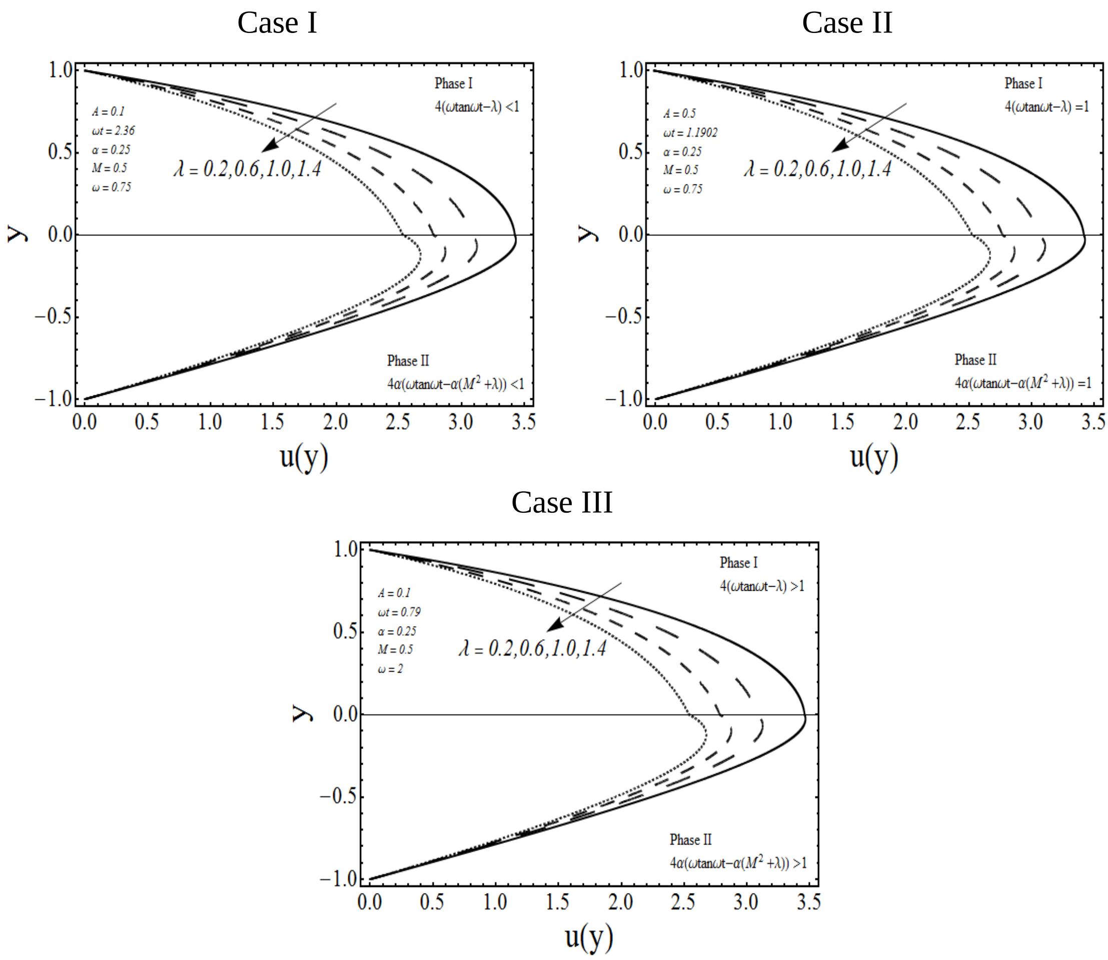

Fig. 4: $u(y)$ for λ.

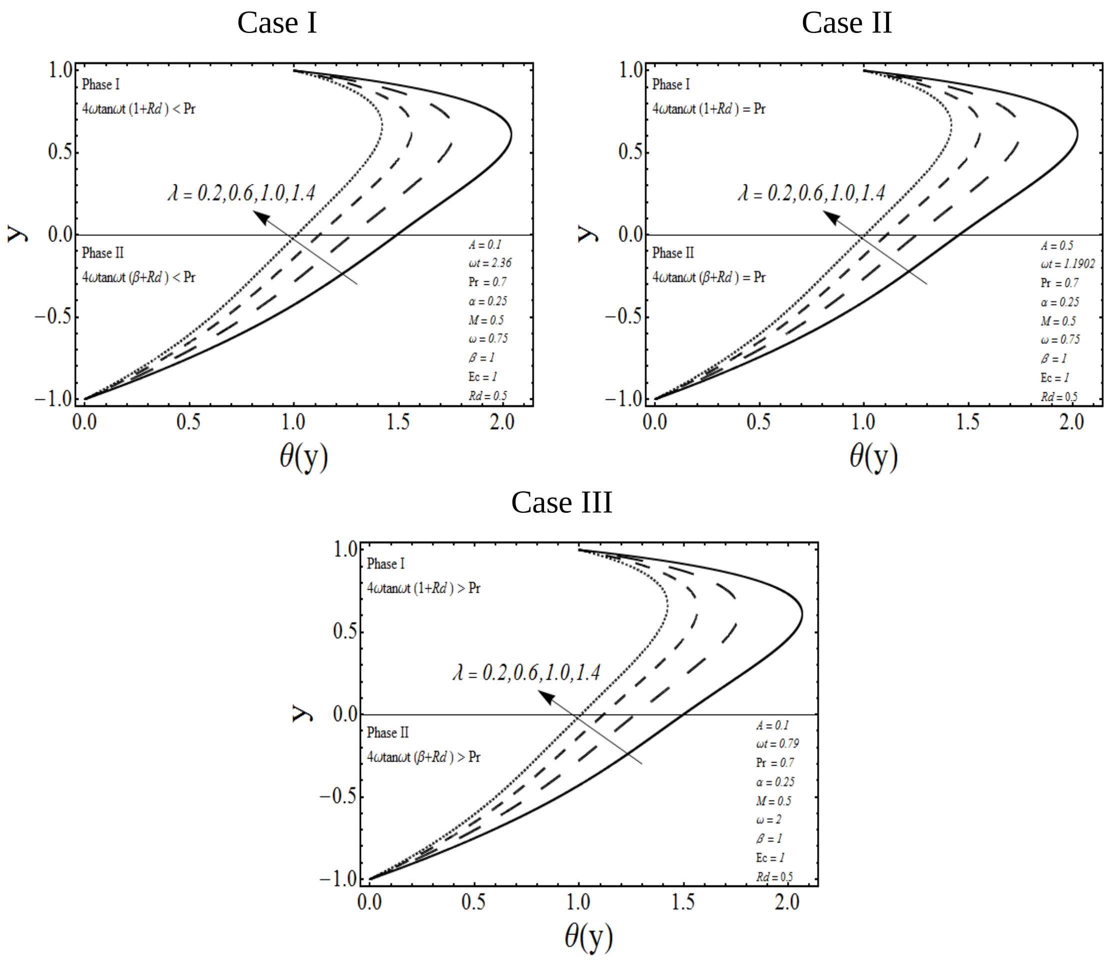

Fig. 5: $\theta\left(y\right)$ for λ.

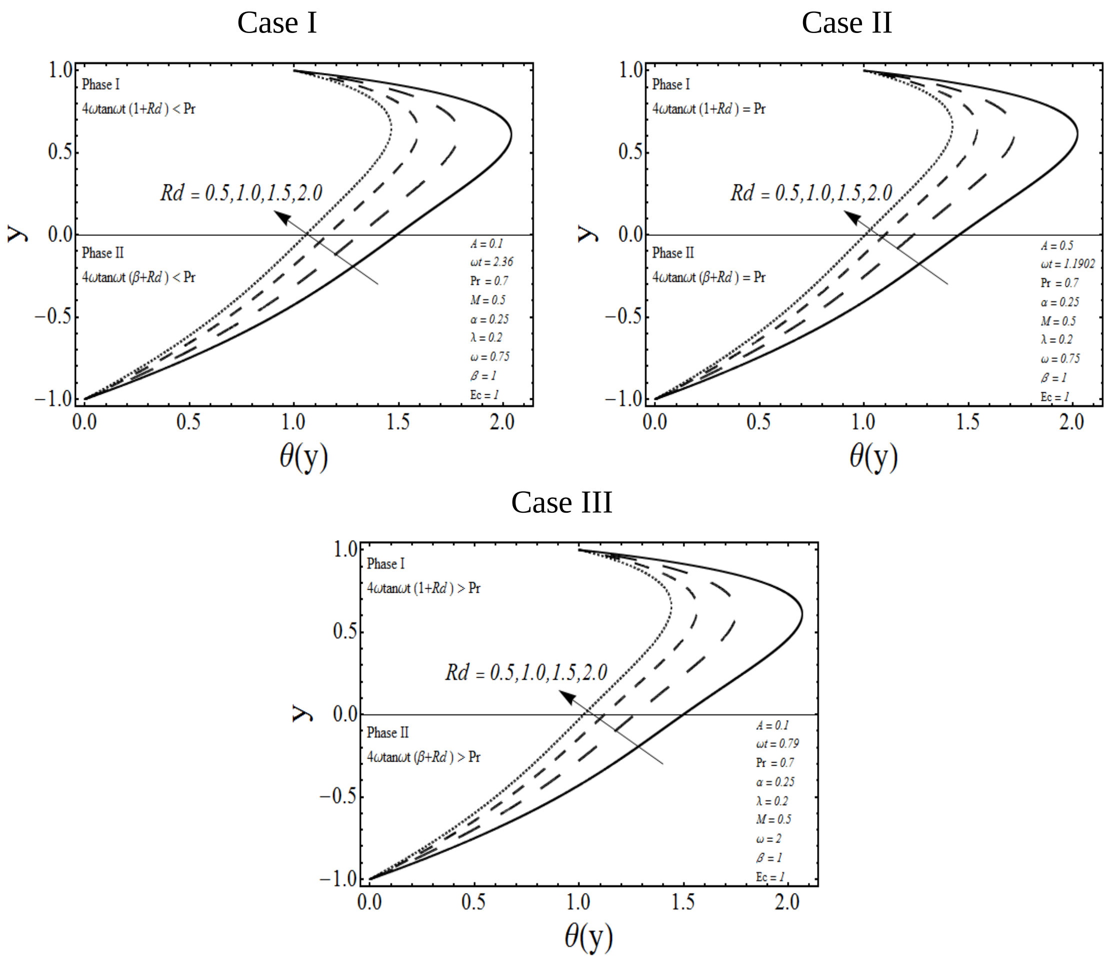

Fig. 6: $\theta(y)$ for Rd.

Table 1: U and θ for εA with $\lambda = 0.2, M = 0.4$, $P = 5$, $\omega t = 2.36$, $\omega = 0.75$, $\Pr = 0.7$, $Ec = 1, \alpha = 0.25$, $\beta = 1$, $Rd = 0.5$.

y	$\varepsilon A = 0.00$ U	θ	$\varepsilon A = 0.10$ U	θ	$\varepsilon A = 0.20$ U	θ
1.00	0.00000	1.00000	0.00000	1.00000	0.00000	1.00000
0.80	1.38304	1.86122	1.36987	2.76852	1.35671	3.67583
0.60	2.34402	2.03199	2.33456	3.81979	2.32510	5.60759
0.40	2.96643	1.90595	2.96974	4.55345	2.97304	7.20095
0.20	3.31614	1.68337	3.33617	5.18783	3.35621	8.69229
0.00	3.44513	1.45265	3.48254	5.83303	3.51994	10.21340
-0.20	3.24375	1.23389	3.32815	5.34818	3.41255	9.46247
-0.40	2.63413	1.00308	2.72938	4.05395	2.82462	7.10483
-0.60	1.83502	0.72989	1.91522	2.67927	1.99541	4.62865
-0.80	0.94284	0.39768	0.98986	1.32944	1.03688	2.26120
-1.00	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000

Table 2: U and θ for ωt with $\lambda = 0.2$, $M = 0.4$, $P = 5$, $\varepsilon A = 0.001$, $\omega = 0.75$, $\Pr = 0.7$, $Ec = 1, \alpha = 0.25$, $\beta = 1$, $Rd = 0.5$.

y	$\omega t = 2.0952$ U	θ	$\omega t = 2.3571$ U	θ	$\omega t = 2.6190$ U	θ
1.00	0.00000	1.00000	0.00000	1.00000	0.00000	1.00000
0.80	1.38293	1.8646	1.38291	1.87021	1.38291	1.88134
0.60	2.34392	2.0387	2.34393	2.04970	2.34396	2.07152
0.40	2.96640	1.91593	2.96646	1.93218	2.96654	1.96432
0.20	3.31622	1.69665	3.31634	1.71809	3.31648	1.76034
0.00	3.44532	1.46939	3.44551	1.49606	3.44570	1.54831
-0.20	3.24424	1.86460	3.24459	1.27466	3.24492	1.32378
-0.40	2.63471	2.03870	2.63508	1.03331	2.63541	1.06998
-0.60	1.83552	1.91593	1.83582	0.74920	1.83609	0.77275
-0.80	0.94315	1.69665	0.94331	0.40691	0.94345	0.41817
-1.00	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000

Table 3: U and θ for εA with $\lambda = 0.2$, $M = 0.4$, $P = 5$, $\omega t = 1.1902$, $\omega = 0.75$, $\Pr = 0.9999$, $Ec = 1$, $\alpha = 0.25$, $\beta = 1$, $Rd = 0.5$.

y	$\varepsilon A = 0.00$ U	θ	$\varepsilon A = 0.10$ U	θ	$\varepsilon A = 0.20$ U	θ
1.00	0.00000	1.00000	0.00000	1.00000	0.00000	1.00000
0.80	1.38304	1.86122	1.36987	2.76852	1.35671	3.67583
0.60	2.34402	2.03199	2.33456	3.81979	2.32510	5.60759
0.40	2.96643	1.90595	2.96974	4.55345	2.97304	7.20095
0.20	3.31614	1.68337	3.33617	5.18783	3.35621	8.69229
0.00	3.44513	1.45265	3.48254	5.83303	3.51994	10.21340
-0.20	3.24375	1.23389	3.32815	5.34818	3.41255	9.46247
-0.40	2.63413	1.00308	2.72938	4.05395	2.82462	7.10483
-0.60	1.83502	0.72989	1.91522	2.67927	1.99541	4.62865
-0.80	0.94284	0.39768	0.98986	1.32944	1.03688	2.26120
-1.00	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000

Table 4: U and θ for ωt with $\lambda = 0.2, M = 0.4$, $P = 5$, $\varepsilon A = 0.005$, $\omega = 0.75$, $\Pr = 0.7$, $Ec = 1, \alpha = 0.25$, $\beta = 1$, $Rd = 0.5$. $\alpha = 0.25$, $\beta = 1$, $Rd = 0.5$.

y	$\omega = 0.0001$, $\omega t = 1.5704$ U	θ	$\omega = 0.1$, $\omega t = 1.1902$ U	θ	$\omega = 1.0$, $\omega t = 0.245$ U	θ	$\omega = 5$, $\omega t = 2.6190$ U	θ
1.00	0.00000	1.00000	0.00000	1.00000	0.00000	1.00000	0.00000	1.00000
0.80	1.38303	2.27983	1.37450	2.27476	1.36074	2.26658	1.35754	2.26435
0.60	2.34402	2.54229	2.33681	2.53457	2.32519	2.52212	2.32249	2.51873
0.40	2.96642	2.37865	2.96032	2.37000	2.95048	2.35607	2.94819	2.35228
0.20	3.31614	2.08501	3.31096	2.07663	3.30262	2.06312	3.30068	2.05945
0.00	3.44513	1.78907	3.44075	1.78174	3.43369	1.76992	3.43205	1.76671
-0.20	3.24375	1.51799	3.24165	1.51196	3.23825	1.50224	3.23746	1.49960
-0.40	2.63413	1.23640	2.63348	1.23176	2.63243	1.22428	2.63218	1.22226
-0.60	1.83503	0.90167	1.83542	0.89853	1.83605	0.89347	1.83620	0.89211
-0.80	0.94285	0.49187	0.94404	0.49029	0.94597	0.48775	0.94642	0.48707
-1.00	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000

Table 5: U and θ for εA with $\lambda = 0.2,\ M = 0.4, P = 5,\ \omega t = 0.7857,\ \omega = 2,\ \Pr = 0.7, Ec = 1,$ $\alpha = 0.25, \beta = 1,\ Rd = 0.5.$

y	$\varepsilon A = 0.00$		$\varepsilon A = 0.10$		$\varepsilon A = 0.20$	
	U	θ	U	θ	U	θ
1.00	0.00000	1.00000	0.00000	1.00000	0.00000	1.00000
0.80	1.38304	1.86122	2.8857	4.80411	4.38837	7.7470
0.60	2.34402	2.03199	4.96469	6.17468	7.58535	10.3174
0.40	2.96643	1.90595	6.32239	6.33351	9.67836	10.7611
0.20	3.31614	1.68337	7.05021	5.96523	10.7843	10.2471
0.00	3.44513	1.45265	7.24344	5.39619	11.0417	9.33972
-0.20	3.24375	1.23389	5.65855	4.70403	8.07335	8.17417
-0.40	2.63413	1.00308	3.96394	3.79306	5.29376	6.58304
-0.60	1.83502	0.72989	2.43861	2.65630	3.04219	4.58270
-0.80	0.94284	0.39768	1.13165	1.36399	1.32046	2.33029
-1.00	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000

Table 6: U and θ for ωt with $\lambda = 0.2, M = 0.4,\ P = 5,\ \varepsilon A = 0.001, \Pr = 0.7,\ Ec = 1,$ $\alpha = 0.25,\ \beta = 1,\ Rd = 0.5.$

y	$\omega t = 0.5238$		$\omega t = 0.7857$		$\omega t = 1.0476$	
	U	θ	U	θ	U	θ
1.00	0.00000	1.00000	0.00000	1.00000	0.00000	1.00000
0.80	1.38333	1.8617	1.38454	1.86416	1.38277	1.85957
0.60	2.34451	2.03263	2.34664	2.03613	2.34355	2.02918
0.40	2.96706	1.90661	2.96978	1.91038	2.96584	1.90238
0.20	3.31685	1.68398	3.31988	1.68765	3.31552	1.67935
0.00	3.44589	1.45319	3.44893	1.45160	3.44456	1.44848
-0.20	3.24395	1.23435	3.24617	1.23736	3.23582	1.22978
-0.40	2.63409	1.00342	2.63546	1.00587	2.62573	0.99905
-0.60	1.83493	0.73011	1.83563	0.73182	1.82943	0.72650
-0.80	0.94278	0.39778	0.94303	0.39865	0.94052	0.39569
-1.00	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000

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Appendix

$$C_1=\frac{-P(m_3(e^{m_4}l_1-l_2e^{m_5})+l_3(m_5e^{m_8}-m_4e^{m_6})+l_4e^{m_7})}{m_3\lambda(l_5e^{m_6}-l_1e^{m_9}+l_2e^{m_{10}}-l_6e^{m_8})},\quad C_2=\frac{1}{e^{m_2}}\left(\frac{P}{\lambda}-C_1e^{m_1}\right),$$

$$C_3=e^{m_4}\left(\frac{P}{\alpha m_3}-C_4e^{-m_5}\right),\quad C_4=\frac{e^{m_5}}{e^{m_5}-e^{m_4}}\left(C_1+C_2-\frac{P}{\lambda}-\frac{P}{\alpha m_3}(e^{m_4}-1)\right),$$

$$C_5=1-l_7-C_6e^{m_{12}},\quad C_6=\frac{l_{10}(1-e^{m_{14}})-\beta m_{14}(1-l_7+l_8+l_9)}{\beta m_{14}(1-e^{m_{12}})+m_{12}(e^{m_{14}}-1)},\quad C_7=-l_8-C_8e^{-m_{14}},\quad C_8=\frac{C_6m_{12}+l_{10}}{\beta m_{14}},$$

$$C_9=\frac{e^{m_{16}}l_{24}-l_{11}l_{23}}{l_{23}e^{m_{15}}-l_{22}e^{m_{16}}},\quad C_{10}=\frac{-C_9e^{m_{15}}-l_{11}}{e^{m_{16}}},\quad C_{11}=\frac{-C_{12}e^{-m_{18}}-l_{12}}{e^{-m_{17}}},\quad C_{12}=\frac{-(C_9+C_{10}+l_{20})}{l_{16}},$$

$$C_{13}=\frac{e^{m_{20}}l_{37}-l_{25}l_{36}}{l_{36}e^{m_{19}}-l_{35}e^{m_{20}}},\quad C_{14}=\frac{-C_{13}e^{m_{19}}-l_{25}}{e^{m_{20}}},\quad C_{15}=\frac{-C_{16}e^{-m_{26}}-l_{26}}{e^{-m_{25}}},\quad C_{16}=\frac{-(C_{13}+C_{14}+l_{33})}{l_{29}},$$

$$D_1=\frac{P_{13}P_2-P_3}{P_1-P_2P_{12}},\quad D_2=\frac{-D_1P_1-P_3}{P_2},\quad D_3=D_1+P_7,\quad D_4=\frac{-D_3P_4-P_6}{P_5},$$

$$D_5=\frac{Q_6Q_{13}-Q_1}{Q_5-Q_6Q_{12}},\quad D_6=\frac{-D_5Q_5-Q_1}{Q_6},\quad D_7=D_5+Q_3,\quad D_8=\frac{-D_7Q_7-Q_2}{Q_8},$$

$$G_1=\frac{R_{13}R_2-R_3\delta_2}{R_1\delta_2-R_2R_{12}},\quad G_2=\frac{-G_1R_1-R_3}{R_2},\quad G_3=G_1+R_7,\quad G_4=\frac{-G_3R_4-R_6}{R_5},$$

$$G_5=\frac{V_{13}V_6-V_1\delta_3}{V_5\delta_3-V_6V_{12}},\quad G_6=\frac{-G_5V_5-V_1}{V_6},\quad G_7=G_5+V_3,\quad G_8=\frac{-G_7V_7-V_2}{V_8}.$$

$$m_1=\frac{1-\sqrt{1+4\lambda}}{2},\quad m_2=\frac{1+\sqrt{1+4\lambda}}{2},\quad m_3=M^2+\lambda,\quad m_4=\frac{1-\sqrt{1+4\alpha^2m_3}}{2},\quad m_5=\frac{1+\sqrt{1+4\alpha^2m_3}}{2},$$

$$m_6=m_2+m_4,\quad m_7=m_6+m_5,\quad m_8=m_2+m_5,\quad m_9=m_1+m_4,\quad m_{10}=m_1+m_5,$$

$$l_1=m_2-\alpha m_4,\quad l_2=m_2-\alpha m_5,\quad l_3=\lambda-\alpha m_3,\quad l_4=\lambda(m_4-m_5),\quad l_5=m_1-\alpha m_4,\quad l_6=m_1-\alpha m_5,$$

$$m_{11}=m_1+m_2,\quad m_{12}=\frac{\Pr}{1+Rd},\quad m_{13}=m_4+m_5,\quad m_{14}=\frac{\Pr}{\beta+Rd},$$

$$k_1=\frac{-m_1m_{12}EcC_1^2}{4m_1-2m_{12}},\quad k_2=\frac{-m_2m_{12}EcC_2^2}{4m_2-2m_{12}},\quad k_3=\frac{-2m_1m_2m_{12}EcC_1C_2}{m_{11}(m_{11}-m_{12})},\quad k_4=\frac{M^2P^2Ec}{m_3^2\alpha},$$

$$k_5=\frac{2m_{14}EcM^2PC_3}{m_3m_4(m_4-m_{14})},\quad k_6=\frac{2m_{14}EcM^2PC_4}{m_3m_5(m_5-m_{14})},\quad k_7=\frac{-m_{14}Ec\alpha C_3^2\left(m_4^2+M^2\right)}{2m_4(2m_4-m_{14})},$$

$$k_8=\frac{-m_{14}Ec\alpha C_4^2\left(m_5^2+M^2\right)}{2m_5(2m_5-m_{14})},\quad k_9=\frac{-2m_{14}Ec\alpha C_3C_4\left(m_4m_5+M^2\right)}{m_{13}(m_{13}-m_{14})},$$

$$l_7=k_1e^{2m_1}+k_2e^{2m_2}+k_3e^{m_{11}},\quad l_8=-k_4+k_5e^{-m_4}+k_6e^{-m_5}+k_7e^{-2m_4}+k_8e^{-2m_5}+k_9e^{-m_{13}},$$

$$l_9=k_1+k_2+k_3-k_5-k_6-k_7-k_8-k_9,$$

$$l_{10}=2m_1k_1+2m_2k_2+m_{11}k_3-\beta(k_4+m_4k_5+m_5k_6+2m_4k_7+2m_5k_8+k_9m_{13}),$$

$$m_{15}=\frac{1+\sqrt{1-4k_{11}}}{2},\quad m_{16}=\frac{1-\sqrt{1-4k_{11}}}{2},\quad m_{17}=\frac{1+\sqrt{1-4\alpha k_{14}}}{2\alpha},\quad m_{18}=\frac{1-\sqrt{1-4\alpha k_{14}}}{2\alpha},$$

$$k_{10}=\omega\tan\omega t,\quad k_{11}=k_{10}-\lambda,\quad k_{12}=\frac{AC_1m_1}{m_1^2-m_1+k_{11}},\quad k_{13}=\frac{AC_2m_2}{m_2^2-m_2+k_{11}},$$

$$k_{14}=k_{10}-\alpha m_3,\quad k_{15}=\frac{AC_3m_4}{\alpha m_4^2-m_4+k_{14}},\quad k_{16}=\frac{AC_4m_5}{\alpha m_5^2-m_5+k_{14}},$$

$$l_{11}=k_{12}e^{m_1}+k_{13}e^{m_2},\quad l_{12}=k_{15}e^{-m_4}+k_{16}e^{-m_5},\quad l_{13}=k_{12}+k_{13}-k_{15}-k_{16},$$

$$l_{14}=m_1k_{12}+m_2k_{13}-\alpha m_4k_{15}-\alpha m_5k_{16},\quad l_{16}=\frac{e^{-m_{18}}}{e^{-m_{17}}}-1,\quad l_{17}=\alpha m_{17}-\alpha m_{18},\quad l_{18}=m_{15}-\alpha m_{17},$$

$$l_{19}=m_{16}-\alpha m_{17},\quad l_{20}=l_{13}+\frac{l_{12}}{e^{-m_{17}}},\quad l_{21}=l_{14}-\alpha m_{17}l_{13},\quad l_{23}=l_{19}-\frac{l_{17}}{l_{16}},\quad l_{24}=l_{21}-\frac{l_{17}l_{20}}{l_{16}},$$

$$m_{19}=\frac{\Pr+\sqrt{\Pr^2-4\Pr k_{10}(1+Rd)}}{2(1+Rd)},\quad m_{20}=\frac{\Pr-\sqrt{\Pr^2-4\Pr k_{10}(1+Rd)}}{2(1+Rd)},$$

$$m_{21}=m_1+m_{15},\quad m_{22}=m_1+m_{16},\quad m_{23}=m_2+m_{15},\quad m_{24}=m_2+m_{16},$$

$$m_{25}=\frac{\Pr+\sqrt{\Pr^2-4\Pr k_{10}(\beta+Rd)}}{2(\beta+Rd)},\quad m_{26}=\frac{\Pr-\sqrt{\Pr^2-4\Pr k_{10}(\beta+Rd)}}{2(\beta+Rd)},$$

$$m_{27}=m_4+m_{17},\quad m_{28}=m_4+m_{18},\quad m_{29}=m_5+m_{17},\quad m_{30}=m_5+m_{18},$$

$$k_{17}=\frac{Am_{12}C_6}{k_{10}},\quad k_{18}=\frac{m_{11}m_{12}Ak_3-2m_1m_2m_{12}Ec(C_2k_{12}+C_1k_{13})}{m_{11}^2-m_{11}m_{12}+m_{12}k_{10}},\quad k_{19}=\frac{2m_1m_{12}(Ak_1-m_1C_1Eck_{12})}{4m_1^2-2m_1m_{12}+m_{12}k_{10}},$$

$$k_{20}=\frac{2m_2m_{12}(Ak_2-m_2C_2Eck_{13})}{4m_2^2-2m_2m_{12}+m_{12}k_{10}},\quad k_{21}=\frac{-2m_1m_{12}m_{15}EcC_1C_9}{m_{21}^2-m_{21}m_{12}+m_{12}k_{10}},\quad k_{22}=\frac{-2m_1m_{12}m_{16}EcC_1C_{10}}{m_{22}^2-m_{22}m_{12}+m_{12}k_{10}},$$

$$k_{23}=\frac{-2m_2m_{12}m_{15}EcC_2C_9}{m_{23}^2-m_{23}m_{12}+m_{12}k_{10}},\quad k_{24}=\frac{-2m_2m_{12}m_{16}EcC_2C_{10}}{m_{24}^2-m_{24}m_{12}+m_{12}k_{10}},\quad k_{25}=\frac{Ak_4}{k_{10}},$$

$$k_{26}=\frac{(Am_3m_4k_5+2k_{15}EcM^2P)m_{14}}{m_3(m_4^2-m_4m_{14}+m_{14}k_{10})},\quad k_{27}=\frac{(Am_3m_5k_5+2k_{16}EcM^2P)m_{14}}{m_3(m_5^2-m_5m_{14}+m_{14}k_{10})},$$

$$k_{28}=\frac{2(Am_4k_7-C_3k_{15}Ec\alpha(m_4^{\,2}+M^2))m_{14}}{4m_4^{\,2}-2m_4m_{14}+m_{14}k_{10}},\quad k_{29}=\frac{2(Am_5k_8-C_3k_{16}Ec\alpha(m_5^{\,2}+M^2))m_{14}}{4m_5^{\,2}-2m_5m_{14}+m_{14}k_{10}},$$

$$k_{30}=\frac{Am_{14}C_8}{k_{10}},\quad k_{31}=\frac{m_{13}m_{14}Ak_9-2m_{14}Ec\alpha(m_4m_5+M^2)(C_4k_{15}+C_3k_{16})}{m_{13}^{\,2}-m_{13}m_{14}+m_{14}k_{10}},$$

$$k_{32}=\frac{2m_{14}C_{11}EcM^2P}{m_3(m_{17}^{\,2}-m_{17}m_{14}+m_{14}k_{10})},\quad k_{33}=\frac{2m_{14}C_{12}EcM^2P}{m_3(m_{18}^{\,2}-m_{18}m_{14}+m_{14}k_{10})},$$

$$k_{34}=\frac{-2m_{14}C_3C_{11}Ec\alpha(m_4m_{17}+M^2)}{m_{27}^{\,2}-m_{27}m_{14}+m_{14}k_{10}},\quad k_{35}=\frac{-2m_{14}C_3C_{12}Ec\alpha(m_4m_{18}+M^2)}{m_{28}^{\,2}-m_{28}m_{14}+m_{14}k_{10}},$$

$$k_{36}=\frac{-2m_{14}C_4C_{11}Ec\alpha(m_5m_{17}+M^2)}{m_{29}^{\,2}-m_{29}m_{14}+m_{14}k_{10}},\quad k_{37}=\frac{-2m_{14}C_4C_{12}Ec\alpha(m_5m_{18}+M^2)}{m_{30}^{\,2}-m_{30}m_{14}+m_{14}k_{10}},$$

$$l_{25}=k_{17}e^{m_{12}}+k_{18}e^{m_{11}}+k_{19}e^{2m_1}+k_{20}e^{2m_2}+k_{21}e^{m_{21}}+k_{22}e^{m_{22}}+k_{23}e^{m_{23}}+k_{24}e^{m_{24}},$$

$$l_{26}=k_{25}+k_{26}e^{-m_4}+k_{27}e^{-m_5}+k_{28}e^{-2m_4}+k_{29}e^{-2m_5}+k_{30}e^{-m_{14}}+k_{31}e^{-m_{13}}$$
$$+k_{32}e^{-m_{17}}+k_{33}e^{-m_{18}}+k_{34}e^{-m_{27}}+k_{35}e^{-m_{28}}+k_{36}e^{-m_{29}}+k_{37}e^{-m_{30}},$$

$$l_{27}=k_{17}+k_{18}+k_{19}+k_{20}+k_{21}+k_{22}+k_{23}+k_{24}-\begin{pmatrix}k_{25}+k_{26}+k_{27}+k_{28}+k_{29}+k_{30}+k_{31}\\+k_{32}+k_{33}+k_{34}+k_{35}+k_{36}+k_{37}\end{pmatrix},$$

$$l_{28}=\begin{pmatrix}m_{12}k_{17}+m_{11}k_{18}+2m_1k_{19}+2m_2k_{20}\\+m_{21}k_{21}+m_{22}k_{22}+m_{23}k_{23}+m_{24}k_{24}\end{pmatrix}-\beta\begin{pmatrix}m_4k_{26}+m_5k_{27}+2m_4k_{28}+2m_5k_{29}+m_{14}k_{30}+m_{13}k_{31}\\+m_{17}k_{32}+m_{18}k_{33}+m_{27}k_{34}+m_{28}k_{35}+m_{29}k_{36}+m_{30}k_{37}\end{pmatrix},$$

$$l_{29}=\frac{e^{-m_{26}}}{e^{-m_{25}}}-1,\quad l_{30}=\beta m_{25}-\beta m_{26},\quad l_{31}=m_{19}-\beta m_{25},\quad l_{32}=m_{20}-\beta m_{25},$$

$$l_{33}=\frac{l_{26}}{e^{-m_{25}}}-1,\quad l_{34}=l_{28}-\beta m_{25}l_{27},\quad l_{35}=l_{31}-\frac{l_{30}}{l_{29}},\quad l_{36}=l_{32}-\frac{l_{30}}{l_{29}},\quad l_{37}=l_{34}-\frac{l_{30}l_{33}}{l_{29}},$$

$$J_1=\frac{1}{2},\quad J_1=\frac{1}{2\alpha},\quad P_1=e^{J_1},\quad P_2=e^{J_1},\quad P_3=k_{12}e^{m_1}+k_{13}e^{m_2},\quad P_4=e^{-J_2},\quad P_5=-e^{-J_2},$$

$$P_6=k_{15}e^{-m_4}+k_{16}e^{-m_5},\quad P_7=k_{12}+k_{13}-k_{15}-k_{16},\quad P_8=m_1k_{12}+m_2k_{13}-\alpha m_4k_{15}-\alpha m_5k_{16},$$

$$P_9=J_1-\alpha J_2,\quad P_{10}=P_8-\alpha P_7J_2,\quad P_{11}=P_6+P_4P_7,\quad P_{12}=P_9+\frac{\alpha P_4}{P_5},\quad P_{13}=P_{10}+\frac{\alpha P_{11}}{P_5},$$

$$J_1=\frac{\Pr}{2(1+Rd)},\quad J_1=\frac{\Pr}{2(\beta+Rd)},\quad n_1=J_1+m_1,\quad n_2=J_1+m_2,\quad n_3=J_2+m_4,\quad n_4=J_2+m_5,$$

$$P_{14}=-2m_1m_{12}C_1Ec\left(\frac{D_2+D_1J_1}{n_1^{\,2}-n_1m_{12}+m_{12}k_{10}}+\frac{(m_{12}-2n_1)D_2J_1}{(n_1^{\,2}-n_1m_{12}+m_{12}k_{10})^2}\right),$$

$$P_{15}=-2m_2m_{12}C_2Ec\left(\frac{D_2+D_1J_1}{n_2{}^2-n_2m_{12}+m_{12}k_{10}}+\frac{\left(m_{12}-2n_2\right)D_2J_1}{\left(n_2{}^2-n_2m_{12}+m_{12}k_{10}\right)^2}\right),$$

$$P_{16}=\frac{-2m_1m_{12}C_1D_2EcJ_1}{n_1{}^2-n_1m_{12}+m_{12}k_{10}},\quad P_{17}=\frac{-2m_2m_{12}C_2D_2EcJ_1}{n_2{}^2-n_2m_{12}+m_{12}k_{10}},$$

$$P_{18}=-2m_4m_{14}C_3Ec\alpha\left(\frac{D_4+D_3J_2}{n_3{}^2-n_3m_{14}+m_{14}k_{10}}+\frac{\left(m_{14}-2n_3\right)D_4J_3}{\left(n_3{}^2-n_3m_{14}+m_{14}k_{10}\right)^2}\right),$$

$$P_{19}=-2m_5m_{14}C_4Ec\alpha\left(\frac{D_4+D_3J_2}{n_4{}^2-n_4m_{14}+m_{14}k_{10}}+\frac{\left(m_{14}-2n_4\right)D_4J_3}{\left(n_4{}^2-n_4m_{14}+m_{14}k_{10}\right)^2}\right),$$

$$P_{20}=\frac{-2m_4m_{14}C_3D_4Ec\alpha J_2}{n_3{}^2-n_3m_{14}+m_{14}k_{10}},\quad P_{21}=\frac{-2m_5m_{14}C_4D_4Ec\alpha J_2}{n_4{}^2-n_4m_{14}+m_{14}k_{10}},$$

$$P_{22}=-2m_{14}C_3EcM^2\alpha\left(\frac{D_3}{n_3{}^2-n_3m_{14}+m_{14}k_{10}}+\frac{\left(m_{14}-2n_3\right)D_4}{\left(n_3{}^2-n_3m_{14}+m_{14}k_{10}\right)^2}\right),$$

$$P_{23}=-2m_{14}C_4EcM^2\alpha\left(\frac{D_3}{n_4{}^2-n_4m_{14}+m_{14}k_{10}}+\frac{\left(m_{14}-2n_4\right)D_4}{\left(n_4{}^2-n_4m_{14}+m_{14}k_{10}\right)^2}\right),$$

$$P_{24}=\frac{-2mm_{14}C_3D_4\mathrm{Ec}M^2}{n_3{}^2-n_3m_{14}+m_{14}k_{10}},\quad P_{25}=\frac{-2mm_{14}C_4D_4\mathrm{Ec}M^2}{n_4{}^2-n_4m_{14}+m_{14}k_{10}},$$

$$P_{26}=\frac{2m_{14}\mathrm{Ec}M^2P}{m_3}\left(\frac{D_3}{J_2^2-J_2m_{14}+m_{14}k_{10}}+\frac{\left(m_{14}-2J_2\right)D_4}{\left(J_2^2-J_2m_{14}+m_{14}k_{10}\right)^2}\right),$$

$$P_{27}=\frac{2m_{14}D_4EcM^2P}{m_3\left(J_2^2-J_2m_{14}+m_{14}k_{10}\right)},\quad P_{28}=P_{18}+P_{22},\quad P_{29}=P_{19}+P_{23},\quad P_{30}=P_{20}+P_{24},$$

$$Q_1=k_{17}e^{m_{12}}+k_{18}e^{m_{11}}+k_{19}e^{2m_1}+k_{20}e^{2m_2}+\left(P_{14}+P_{16}\right)e^{n_1}+\left(P_{15}+P_{17}\right)e^{n_2},$$

$$P_{31}=P_{21}+P_{25},\quad Q_2=k_{25}+k_{26}e^{-m_4}+k_{27}e^{-m_5}+k_{28}e^{-2m_4}+k_{29}e^{-2m_5}+k_{30}e^{-m_{14}}+k_{31}e^{-m_{13}}$$
$$+\left(P_{26}-P_{27}\right)e^{-J_2}+\left(P_{28}-P_{30}\right)e^{-n_3}+\left(P_{29}-P_{31}\right)e^{-n_4},$$

$$Q_3=k_{17}+k_{18}+k_{19}+k_{20}+P_{14}+P_{15}-\begin{pmatrix}k_{25}+k_{26}+k_{27}+k_{28}+k_{29}+k_{30}\\+k_{31}+P_{26}+P_{28}+P_{29}\end{pmatrix},$$

$$Q_4=\begin{pmatrix}m_{12}k_{17}+m_{11}k_{18}+2m_1k_{19}+2m_2k_{20}\\+n_1P_{14}+n_2P_{15}+P_{16}+P_{17}\end{pmatrix}-\beta\begin{pmatrix}m_4k_{26}+m_5k_{27}+2m_4k_{28}+2m_5k_{29}+m_{14}k_{30}+m_{13}k_{31}\\+J_2P_{26}+P_{27}+n_3P_{28}+n_4P_{29}+P_{30}+P_{31}\end{pmatrix},$$

$$Q_5 = e^{J_3}, \quad Q_6 = e^{J_3}, \quad Q_7 = e^{-J_4}, \quad Q_8 = -e^{-J_4}, \quad Q_9 = J_3 - \beta J_4,$$

$$Q_{10} = Q_4 - \beta J_4 Q_3, \quad Q_{11} = Q_7 Q_3 + Q_2, \quad Q_{12} = Q_9 + \frac{\beta Q_7}{Q_8}, \quad Q_{13} = Q_{10} + \frac{\beta Q_{11}}{Q_8},$$

$$\gamma_1 = \frac{1}{2}, \ \gamma_2 = \frac{1}{2\alpha}, \quad \delta_1 = \frac{\sqrt{4(\omega \tan \omega t - \lambda) - 1}}{2}, \quad \delta_2 = \frac{\sqrt{4\alpha(\omega \tan \omega t - \alpha m_3) - 1}}{2\alpha},$$

$$R_1 = e^{\gamma_1} \cos \delta_1, \quad R_2 = e^{\gamma_1} \sin \delta_1, \quad R_3 = k_{12} e^{m_1} + k_{13} e^{m_2}, \quad R_4 = e^{-\gamma_2} \cos \delta_2, \quad R_5 = e^{-\gamma_1} \sin \delta_2,$$

$$R_6 = k_{15} e^{-m_4} + k_{16} e^{-m_5}, \ R_7 = k_{12} + k_{13} - k_{15} - k_{16}, \ R_8 = m_1 k_{12} + m_2 k_{13} - \alpha m_4 k_{15} - \alpha m_5 k_{16},$$

$$R_9 = \gamma_1 - \alpha \gamma_2, \ R_{10} = R_8 - \alpha R_7 \gamma_2, \ R_{11} = R_6 + R_4 R_7, \ R_{12} = R_9 - \frac{\alpha \delta_2 R_4}{R_5}, \ R_{13} = R_{10} - \frac{\alpha \delta_2 R_{11}}{R_5},$$

$$\gamma_3 = \frac{\Pr}{2(1+Rd)}, \ \gamma_4 = \frac{\Pr}{2(\beta + Rd)}, \ \delta_3 = \frac{\sqrt{4 \Pr \omega \tan \omega t (1+Rd) - \Pr^2}}{2(1+Rd)},$$

$$\delta_4 = \frac{\sqrt{4 \Pr \omega \tan \omega t (\beta + Rd) - \Pr^2}}{2(\beta + Rd)}, \quad w_1 = \gamma_1 + m_1, \ w_2 = \gamma_1 + m_2, \ w_3 = \gamma_2 + m_4, \ w_4 = \gamma_2 + m_5,$$

$$S_1 = w_1^2 - w_1 m_{12} + m_{12} k_{10} - \delta_1^2, \quad S_2 = 2w_1 - m_{12}, \ S_3 = w_2^2 - w_2 m_{12} + m_{12} k_{10} - \delta_1^2, \quad S_4 = 2w_2 - m_{12},$$

$$S_5 = w_3^2 - w_3 m_{14} + m_{14} k_{10} - \delta_2^2, \quad S_6 = 2w_3 - m_{14}, \ S_7 = w_4^2 - w_4 m_{14} + m_{14} k_{10} - \delta_2^2, \quad S_8 = 2w_4 - m_{14},$$

$$S_9 = \gamma_2^2 - \gamma_2 m_{14} + m_{14} k_{10} - \delta_2^2, \quad S_{10} = 2\gamma_2 - m_{14},$$

$$R_{14} = G_1 \gamma_1 - G_2 \delta_1, \ R_{15} = G_2 \gamma_1 - G_1 \delta_1, \ R_{16} = \frac{-2 m_1 m_{12} C_1 Ec}{S_1^2 + S_2^2 \delta_1^2}, \ R_{17} = \frac{-2 m_2 m_{12} C_2 Ec}{S_3^2 + S_4^2 \delta_1^2},$$

$$R_{18} = R_{14} S_1 - R_{15} S_2 \delta_1, \quad R_{19} = R_{15} S_1 + R_{14} S_2 \delta_1, \ R_{20} = R_{14} S_3 - R_{15} S_4 \delta_1, \quad R_{21} = R_{15} S_3 + R_{14} 4_2 \delta_1,$$

$$R_{22} = G_3 \gamma_2 - G_4 \delta_2, \ R_{23} = G_4 \gamma_2 - G_3 \delta_2, \ R_{24} = \frac{-2 m_4 m_{14} C_3 Ec\alpha}{S_5^2 + S_6^2 \delta_1^2}, \ R_{25} = \frac{-2 m_5 m_{14} C_4 Ec\alpha}{S_7^2 + S_8^2 \delta_1^2},$$

$$R_{26} = R_{22} S_5 - R_{23} S_6 \delta_2, \quad R_{27} = R_{23} S_5 + R_{22} S_6 \delta_2, \ R_{28} = R_{22} S_7 - R_{23} S_8 \delta_2, \quad R_{29} = R_{23} S_7 + R_{22} S_8 \delta_2,$$

$$R_{30} = \frac{2 m_{14} Ec M^2 P}{m_3 \left(S_9^2 + S_{10}^2 \delta_2^2\right)}, \ R_{31} = \frac{-2 m_{14} C_3 Ec M^2 \alpha}{S_5^2 + S_6^2 \delta_2^2}, \quad R_{32} = \frac{-2 m_{14} C_4 Ec M^2 \alpha}{S_7^2 + S_8^2 \delta_2^2},$$

$$R_{33} = G_3 S_5 - G_4 S_6 \delta_2, \quad R_{34} = G_4 S_5 + G_3 S_6 \delta_2, \ R_{35} = G_3 S_7 - G_4 S_8 \delta_2, \quad R_{36} = G_4 S_7 + G_3 S_8 \delta_2,$$

$$R_{37} = G_3 S_9 - G_4 S_{10} \delta_2, \quad R_{38} = G_4 S_9 + G_3 S_{10} \delta_2, \quad R_{39} = R_{24} R_{26} + R_{31} R_{33},$$

$$R_{40} = R_{24} R_{27} + R_{31} R_{34}, \quad R_{41} = R_{25} R_{28} + R_{32} R_{35}, \quad R_{42} = R_{25} R_{29} + R_{32} R_{36},$$

$$V_1 = k_{17} e^{m_{12}} + k_{18} e^{m_{11}} + k_{19} e^{2m_1} + k_{20} e^{2m_2} + R_{16} \left(R_{18} \cos \delta_1 + R_{18} \sin \delta_1\right) e^{w_1} + R_{17} \left(R_{20} \cos \delta_1 + R_{21} \sin \delta_1\right) e^{w_2},$$

$$V_2 = k_{25} + k_{26} e^{-m_4} + k_{27} e^{-m_5} + k_{28} e^{-2m_4} + k_{29} e^{-2m_5} + k_{30} e^{-m_{14}} + k_{31} e^{-m_{13}} + R_{30} \left(R_{37} \cos \delta_2 - R_{38} \sin \delta_2\right) e^{-\gamma_2}$$
$$+ \left(R_{39} \cos \delta_2 - R_{40} \sin \delta_2\right) e^{-w_3} + \left(R_{41} \cos \delta_2 - R_{42} \sin \delta_2\right) e^{-w_4},$$

$$V_3 = k_{17} + k_{18} + k_{19} + k_{20} + R_{16} R_{18} + R_{17} R_{20} - \left(k_{25} + k_{26} + k_{27} + k_{28} + k_{29} + k_{30} + k_{31} + R_{30} R_{37} + R_{39} + R_{41}\right),$$

$$V_4=\begin{pmatrix} m_{12}k_{17}+m_{11}k_{18}+2m_1k_{19} \\ +2m_2k_{20}+w_1R_{16}R_{18}+w_2R_{17}R_{20} \\ +R_{16}R_{19}\delta_1+R_{17}R_{21} \end{pmatrix}-\beta\begin{pmatrix} m_4k_{26}+m_5k_{27}+2m_4k_{28}+2m_5k_{29} \\ +m_{14}k_{30}+m_{13}k_{31}+w_3R_{39}+w_4R_{41} \\ +R_{30}R_{37}\gamma_2+R_{30}R_{38}\delta_2+R_{42}\delta_2 \end{pmatrix},$$

$$V_5=e^{\gamma_3}\cos\delta_3,\quad V_6=e^{\gamma_3}\sin\delta_3,\quad V_7=e^{-\gamma_4}\cos\delta_4,\quad V_8=e^{-\gamma_4}\sin\delta_4,\quad V_9=\gamma_3-\beta\gamma_4,$$

$$V_{10}=V_4-\beta\gamma_4V_3,\quad V_{11}=V_2+V_3V_7,\quad V_{12}=V_9-\frac{\beta\delta_4V_7}{V_8},\quad V_{13}=V_{10}-\frac{\beta\delta_4V_{11}}{V_8},$$